UDC 532.5

SMALL PERTURBATIONS OF A STABLE SURFACE OF CAPILLARY FLUID

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The problem of perturbation of the equilibrium state of a capillary fluid is considered for small variations of physical parameters. Consideration is primarily given to perturbations of a stable spherical surface by a weak gravitational field (the axisymmetric case) and, also, to the three-dimensional problems of perturbation of a free surface which represents part of a horizontal plane (for varying wetting angle). In the latter case the perturbed surface of fluid comprised in a dihedral angle between two semi-infinite vertical planes and contained in cylinders of rectangular and triangular cross sections is determined.

1. Formulation of problem. Let Γ be the free surface of a capillary fluid in a container and γ be the line of fluid contact with the latter. The following conditions must be satisfied in the equilibrium state (see, e.g. [1]):

$$k_1 + k_2 = \sigma^{-1} \Pi + C \quad \text{on } \Gamma, \quad \mathbf{n} \cdot \mathbf{n}_1 = \cos\alpha \quad \text{on } \gamma, \int_\Omega d\Omega = v$$

where k_1 and k_2 are the radii of principal curvature of surface Γ, σ is the coefficient of surface tension, Π is the volume density of the fluid potential energy (assumed to be a known function of coordinates), c is an a priori unknown constant, $\mathbf{n}$ and $\mathbf{n}_1$ are unit vectors of normal to surface Γ and to the container surface, respectively (see Fig. 1), α is the angle of wetting, Ω is the part of space occupaied by the fluid, and v is the fluid volume.

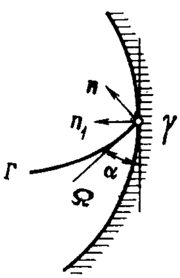

Fig. 1

Let us assume that for certain specific Π (x), α and v the stable surface Γ (hence also the constant c) is known. The problem is to define perturbations of the steady surface induced by specified small increments $\delta\Pi$ (x), $\delta\alpha$ and δv.

We assume that every point $x \in \Gamma$ is subjected to a small displacement $\delta x = \mathbf{h}(x)$ to which corresponds deviation $N = \mathbf{h} \cdot \mathbf{n}$ along the normal. By varying the defined above equilibrium conditions, as in [1], we obtain for the deviation of N the following problem:

$$\Delta N - aN = \sigma^{-1}\delta\Pi + \delta c \quad \text{on} \quad \Gamma \tag{1.1}$$

$$\chi N + \frac{\partial N}{\partial \nu} = -\,\delta\alpha \quad \text{on } \gamma, \int_\Gamma N d\Gamma = \delta v \tag{1.2}$$

$$a = \sigma^{-1}\frac{\partial \Pi}{\partial n} - k_1^2 - k_2^2, \quad \chi = \frac{k\cos\alpha - k_0}{\sin\alpha} \tag{1.3}$$

where Δ is the Laplace-Beltrami operator on Γ, k and k_0 are curvatures of cross sections of Γ and of the container, respectively, by a plane normal to γ (the orientation of these cross sections is shown in Fig. 1), and $\boldsymbol{\nu}$ is the outward normal to γ in the plane normal to Γ.

The problem (1.1), (1.2) in N and δc is of the Fredholm kind. In what follows we assume that the problem has a unique solution for any $\delta\Pi$, $\delta\alpha$ and δv (i.e. the related homogeneous problem does not have nontrivial solutions). This implies that the initially stable surface Γ is not critical in the sense of stability and that perturbations do not lead to the branching of equilibrium states.

Note that Newton's method for the determination of the stable surface of a capillary fluid for specified physical parameters reduces to the sulition of problem (1.1), (1.2).

2. Perturbations of axisymmetric and plane equilibrium states. Let the container, the initial surface Γ, and potential Π be symmetric about the z-axis, and r, θ and z be the related cylindrical coordinates. It is convenient to take the length of arc s of the intersection of surface Γ with the half-plane $\theta=\text{const}$ and angle θ for curvilinear coordinates on Γ Then for $N = N(s, \theta)$ problem (1.1), (1.2) assumes the form

$$\frac{1}{r}\frac{\partial}{\partial s}\left(r\frac{\partial N}{\partial s}\right)+\frac{1}{r^2}\frac{\partial^2 N}{\partial\theta^2}-aN=\sigma^{-1}\delta\Pi+\delta c$$

$$\left(\chi N-\frac{\partial N}{\partial s}\right)\Big|_{s=0}=-\,\delta\alpha,\quad \left(\chi N+\frac{\partial N}{\partial s}\right)\Big|_{s=s_1}=-\,\delta\alpha$$

$$\int_0^{s_1} r\left(\int_0^{2\pi} N d\theta\right) ds=\delta v$$

where $r(s)$ and $a(s)$ are known functions. The solution of $N(s, \theta)$ is readily derived in the form of trigonometric series in θ. Barniak had obtained Green's function for this problem (*).

If $\delta\Pi$ is independent of θ the problem for $N = N(s)$ is simplified

$$N''+\frac{r'}{r}N'-aN=\sigma^{-1}\delta\Pi+\delta c \tag{2.1}$$

$$(\chi N-N')|_{s=0}=-\,\delta\alpha,\quad (\chi N+N')|_{s=s_1}=-\,\delta\alpha,\quad 2\pi\int_0^{s_1} rN ds=\delta v \tag{2.2}$$

Note that for positive $r(0)$ and $r(s_1)$ the conditions at the initial point $s = 0$ and the end points $s = s_1$ of surface Γ generatrix must be satisfied. If, however, point $s = 0$ or $s = s_1$ lies on the axis of rotation, the corresponding boundary condition must be replaced by the condition of boundedness of function $N(s)$.

For a plane problem (on obvious assumptions) instead of (2.1) and (2.2) we have

$$N''-aN=\sigma^{-1}\delta\Pi+\delta c$$

$$(\chi N-N')|_{s=0}=-\delta\alpha,\quad (\chi N+N')|_{s=s_1}=-\,\delta\alpha,\quad \int_0^{s_1} N ds=\delta v$$

where s is the length of arc of cross section of Γ.

Let us pass to the solution of some specific problems.

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*) Barniak, M. Ia., Approximate methods for solving problems of statics and dynamics of fluid in containers under conditions of near-weightlessness. Candidate's dissertation, Kiev, 1971.

3. Stable surfaces for small Bond numbers. Let us determine the shape of a simply-connected axisymmetric surface of fluid in a homogeneous gravitational field of intensity ηg acting in the direction of the z-axis, at small Bond numbers $B = \rho\eta g l^2\sigma^{-1}$ (ρ is the density of fluid, g is the acceleration of gravity, η is the overload coefficient, and l is a characteristic linear dimension). The unknown surface is close to a spherical one, obtainable for the same α and v under conditions of weightlessness ($\Pi \equiv 0$). We take this sphere as the unperturbed surface Γ, and assume that its position in the container and radius R are known (see, e.g. [2]). We direct the z-axis from the liquid to gas and locate the coordinate origin at the point of intersection of that axis with Γ. By selecting R as the characteristic linear dimension (here R is assumed finite; the case of $R = \infty$ and the sphere degenerates into a plane is considered in Sect. 4), problem (2.1),(2.2) in dimensionless form is defined by

$$N'' + \operatorname{ctg} s\, N' + 2N = \pm B\,(1 - \cos s) + \delta c \tag{3.1}$$

$$(\chi N + N')|_{s=s_1} = 0, \quad \int_0^{s_1} N \sin s\, ds = 0, \quad \chi = \pm \frac{\cos\alpha - k_0}{\sin\alpha} \tag{3.2}$$

In these equations the sign plus or minus depends on whether the coordinate z of the sphere center is positive or negative.

The general finite solution of Eq. (3.1) is of the form

$$N = c_1 \cos s \pm B\,[{}^1/_6 + {}^1/_3 \cos s \ln(1 + \cos s)] + {}^1/_2\delta c$$

where constants c_1 and δc are determined by conditions (3.2). Solution of the considered problem was derived in [3].

By selecting as the unperturbed surface the circular cyclinder

$$x = R \sin(s / R), \ z = \pm R\,[1 - \cos(s / R)]$$

in a similar plane problem we obtain in dimensionless coordinates

$$N = c_1 \sin s + c_2 \cos s + \delta c \pm B\,(1 - {}^1/_2\, s \sin s)$$

where R is a characteristic dimension.

4. Surface of small slope. Surfaces of the form $z=\mathrm{const}$ satisfy in a gravitational field of any intensity the equation of equilibrium. Let us assume that the plane $z = 0$ is the equilibrium surface in some container for $\alpha = \alpha_0$. We have to determine the perturbation induced by the given variation $\delta\alpha$ of the wetting angle. Then

$$N = z, \quad a = \rho\eta g\sigma^{-1} \equiv b, \quad \chi = -k_0 / \sin\alpha_0, \quad \delta\Pi = \delta v = 0$$

In the axisymmetric case $r(s) \equiv s$ and the general solution of Eq. (2.1) is

$$z = c_1 + c_2 \ln r + {}^1/_2\,\delta c \quad \text{for} \quad b = 0$$
$$z = c_1 I_0(\sqrt{b}r) + c_2 K_0(\sqrt{b}r) - \delta c / b \ \text{for} \ b > 0$$
$$z = c_1 J_0(\sqrt{|b|}r) + c_2 N_0(\sqrt{|b|}r) - \delta c / b \ \text{for} \ b < 0$$

where $J_0(\tau)$ and $I_0(\tau)$ are Bessel functions of a real and imaginary argument of zero order, and $K_0(\tau)$ and $N_0(\tau)$ are Macdonald and Neuman functions.

Since in the plane problem $s \equiv x$ and $N = z(x)$, hence

$$z = c_1 + c_2 s + {}^1/_2\, s^2\, \delta c,\ b = 0$$
$$z = c_1 e^{\sqrt{b}x} + c_2 e^{-\sqrt{b}x} - \delta c\ /\ b,\ \ b > 0$$
$$z = c_1 \sin(\sqrt{|b|}\,x) + c_2 \cos(\sqrt{|b|}\,x) - \delta\, c\ /\ b,\ b < 0$$

The related problems for a fluid contained in a circular cylinder and, also, between parallel vertical plates were solved in [2].

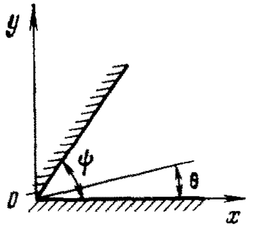

Fig. 2

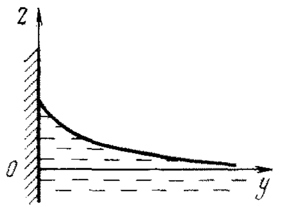

Fig. 3

If $b > 0$ and the fluid is contained within the dihedral angle ψ formed by two semi-infinite vertical plates, then in the unperturbed state $\alpha = \pi\ /\ 2$ and at the free surface $z \equiv 0$.

We superpose the z-axis with the edge of the dihedral angle and locate the x- and y-axes as shown in Fig. 2. In the cylindrical system of coordinates $r,\ \theta,\ z$ we read angle θ from the half-plane xz.

Owing to the linearity of the problem it is sufficient to consider the case in which $\delta\alpha = -1$. Since Γ is unbounded, we reject in (1.2) the second condition, and stipulate that the solution must tend to zero with infinite recession from the plates; consequently in Eq. (1.1) we have $\delta c = 0$. Taking the above into consideration and substituting z for N, we obtain the boundary value problem

$$\Delta z - bz = 0 \quad \text{on} \quad \Gamma, \qquad \partial z\ /\ \partial \nu = 1 \quad \text{on}\ \gamma. \tag{4.1}$$

Behavior of the fluid surface close to the edge. Asymptotics of low gravitation. Before analyzing the solution of problem (4.1), we point out that the height of fluid at the dihedral angle edge $N|_{r=0}$ can be readily determined by using Green's formula

$$\int_\Gamma (vLu - uLv)\, d\Gamma = \int_\gamma \left(v\, \frac{\partial u}{\partial \nu} - u\, \frac{\partial v}{\partial \nu}\right) d\gamma, \quad Lu \equiv \Delta u - bu$$

Setting

$$u = z(r), \quad \left.\frac{\partial u}{\partial \nu}\right|_\gamma = 1, \quad v = K_0(\sqrt{b}\,|\,\mathbf{r} - \mathbf{r}'\,|)$$

where $K_0(\sqrt{b}\,|\,\mathbf{r} - \mathbf{r}'\,|)$ is the Macdonald function which is the functional solution for the equation $Lu = 0$, we obtain

$$\omega z(\mathbf{r}) = \int_\gamma \left[K_0(\sqrt{b}\,|\,\mathbf{r} - \mathbf{r}'\,|) - z(\mathbf{r}')\frac{\partial}{\partial \nu} K_0(\sqrt{b}\,|\,\mathbf{r} - \mathbf{r}'\,|)\right] d\gamma, \quad \mathbf{r}' \in \gamma$$

where $\omega = 2\pi$ for $\mathbf{r} \in \Gamma$, $\omega = \pi$ for $\mathbf{r} \in \gamma\ (\mathbf{r} \neq 0)$ and $\omega = \psi$ for $\mathbf{r} = 0$. For $\mathbf{r} = 0$ we have $\partial K_0\ /\ \partial \nu\,|_\gamma = 0$, hence the height of fluid lift at the dihedral angle edge is

$$z|_{r=0} = \frac{2}{\sqrt{b}\psi} \int_0^\infty K_0(\tau)\, d\tau = \frac{1}{\sqrt{b}}\, \frac{\pi}{\psi} \tag{4.2}$$

Note that it is possible in problem (4.1) to obtain $b = 1$ (by assumption $b > 0$) by passing to dimensionless variables $x' = \sqrt{b}x$, $y' = \sqrt{b}y$ and $z' = \sqrt{b}z$. With these we determine the asymptotic behavior of the solution of problem (4.1) for small $\sqrt{b}\,r$. Expanding function $z'(x', y')$ into a Taylor series, then reverting to variables x, y, z and taking into account (4.2), we obtain

$$z(x, y; b) = \frac{\pi}{\sqrt{b}\psi} + c_1 x + c_2 y + O(\sqrt{b}r^2), \quad z(r, \theta; b) =$$
$$\frac{\pi}{\sqrt{b}\psi} + (c_1 \cos\theta + c_2 \sin\theta) r + O(\sqrt{b}r^2)$$

where the constants c_1 and c_2 are independent of b and are determined by the boundary condition $\partial z / \partial \nu \,|_\gamma = 1$

$$1 = \lim_{r \to 0} \left(-\frac{1}{r}\frac{\partial z}{\partial \theta}\right)\Big|_{\theta=0} = -c_2, \quad 1 = \lim_{r \to 0} \left(\frac{1}{r}\frac{\partial z}{\partial \theta}\right)\Big|_{\theta=\psi} =$$
$$-c_1 \sin\psi + c_2 \cos\psi, \quad c_1 = -\operatorname{ctg}\frac{\psi}{2}$$

Thus

$$z(r, \theta; b) = \frac{1}{\sqrt{b}}\frac{\pi}{\psi} - \left(\operatorname{ctg}\frac{\psi}{2}\cos\theta + \sin\theta\right) r + O(\sqrt{b}r^2)$$

This formula defines the asymptotics of the whole surface of fluid for $b \to +0$, as well as its behavior in the vicinity of the dihedral angle edge for any finite $b > 0$.

In what follows we assume $b = 1$ and omit the prime at variables. Thus we have to consider the boundary value problem

$$\Delta z - z = 0 \quad \text{on} \quad \Gamma, \quad \partial z / \partial \nu = 1 \quad \text{on} \quad \gamma \tag{4.3}$$

Angles $\psi = \pi / n$ and other particular cases. Function

$$e^{-x} = e^{-r\cos\theta}, \quad e^{-y} = e^{-r\sin\theta}, \quad e^{-r\cos(\theta-\delta)} = e^{-(x\cos\delta + y\sin\delta)} \tag{4.4}$$

where δ is an arbitrary constant, evidently satisfies the equation of problem (4.3), and, if $\psi = \pi$, $z = e^{-y} = e^{-r\sin\theta}$ is the solution of the boundary value problem (4.3), i.e. the xz-plane is the wall (Fig. 3).

If $\psi = \pi / 2$, then

$$z = e^{-y} + e^{-x} = e^{-r\sin\theta} + e^{-r\sin(\theta + \pi/2)}$$

For a cylindrical container of rectangular cross section with vertexes at $\pm a_x$ and $\pm a_y$

$$z = \operatorname{ch} x / \operatorname{sh} a_x + \operatorname{ch} y / \operatorname{sh} a_y$$

Function of the form (4.4) can be used for deriving solutions of other problems by applying the method of reflection. Thus for a dihedral angle $\psi = \pi / n$ $(n = 1, 2, 3, \ldots)$ the solution of problem (4.3) is given by formula

$$z = \sum_{k=0}^{n-1} \exp\left[-r \sin\left(\theta + \frac{k\pi}{n}\right)\right] \tag{4.5}$$

which is easily verified by direct substitution into (4.3). For a cylinder whose cross section is an equilateral triangle of height H, and with the coordinate origin within its perimeter

$$z = (1 - e^{-H})^{-1} \sum_{i=1}^{6} \exp(\alpha_i x + \beta_i y - p_i)$$

where $\alpha_i x + \beta_i y - p_i = 0$ ($\alpha_i^2 + \beta_i^2 = 1$, $p_i > 0$ and $i = 1,2, \ldots, 6$) are equations of the triangle sides and of straight lines passing through its vertexes and parallel to the sides.

Solution for arbitrary angles $0 < \psi \leqslant 2\pi$. Passing to polar coordinates r, θ ($0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \psi$) we reduce problem (4.3) to the form

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{d^2 z}{\partial \theta^2} - z = 0, \quad \frac{1}{r}\frac{\partial z}{\partial \theta}\bigg|_{\theta=0} = -1, \quad \frac{1}{r}\frac{\partial z}{\partial \theta}\bigg|_{\theta=\psi} = 1 \qquad (4.6)$$

The derived above results show that for $\psi = \pi / n$ the basic qualitative properties of the sought solution is the exponential decrease along radial half-lines (except boundary half-lines $\theta = 0$ and $\theta = \psi$) for considerable r, and the symmetry with respect to the half-plane $\theta = \psi / 2$. These properties are possesed, for instance, by the following particular solutions of equations of problem (4.6):

$$z = \operatorname{ch}\,[\tau\,(\theta - \psi\,/\,2)]\; K_{i\tau}(r)$$

where $\tau > 0$ is an arbitrary constant and $K_{i\tau}(r)$ is the Macdonald function.

We seek $z(r, \theta)$ in the form of integral

$$z(r, \theta) = \int_0^\infty \varphi(\tau)\operatorname{ch}\,[\tau\,(\theta - \psi/2)]\, K_{i\tau}(r)\, d\tau \qquad (4.7)$$

This ensures that the equation of problem (4.6) is satisfied. We select function $\varphi(\tau)$ so as to have one of the boundary conditions satisfied (the second will be satisfied owing to symmetry). The substitution of formula (4.7) into the boundary condition for $\theta = \psi$ yields

$$\int_0^\infty \varphi(\tau)\,\tau\,\operatorname{sh}\left(\frac{\tau\psi}{2}\right) K_{i\tau}(r)\, d\tau = r \qquad (4.8)$$

For determining from this the function $\varphi(\tau)$ it is expedient to use the Kontorovich-Lebedev integral transformation [4]

$$F(\tau) = \frac{2}{\pi^2}\,\tau\operatorname{sh}(\pi\tau)\int_0^\infty f(r)\, K_{i\tau}(r)\frac{dr}{r}, \quad \int_0^\infty F(\tau)\, K_{i\tau}(r)\, d\tau = f(r) \qquad (4.9)$$

The equality (4.8) can be made identical to the second of formulas (4.9) by setting

$$f(r) = r, \quad F(\tau) = \varphi(\tau)\tau\operatorname{sh}(\tau\psi\,/\,2)$$

Then, using the first of formulas (4.9), we obtain

$$\varphi(\tau) = \frac{2}{\pi^2}\frac{\operatorname{sh}(\pi\tau)}{\operatorname{sh}(\psi\tau/2)}\int_0^\infty K_{i\tau}(r)\, dr = \frac{2}{\pi}\frac{\operatorname{sh}(\pi\tau/2)}{\operatorname{sh}(\psi\tau/2)}$$

Hence the solution of the problem considered here is of the form

$$z(r, \theta) = \frac{2}{\pi}\int_0^\infty \Lambda_z(\tau, \psi, \theta)\, K_{i\tau}(r)\, d\tau \qquad (4.10)$$

$$\Lambda_z(\tau, \psi, \theta) = \operatorname{sh}(\pi\tau\,/\,2)\operatorname{ch}\,[\tau\,(\theta - \psi\,/\,2)]\,/\operatorname{sh}(\psi\tau\,/\,2)$$

where the integral in the right-hand part converges only for $0 < \theta < \psi$, since for $\tau \to \infty$

$$K_{i\tau}(r) \sim \sqrt{\frac{\pi}{2\tau}}\, e^{-\pi\tau/2}$$

At the γ-boundary the values of function $z(r, \theta)$ and of its normal derivative can be obtained by passing to limits $\theta \to 0$ or $\theta \to \psi$ only after integration with respect to τ. For $0 < \psi \leqslant \pi$ this can be avoided by setting

$$z(r, \theta) = e^{-r \sin\theta} + e^{-r \sin(\psi - \theta)} + u(r, \theta)$$

Applying to $u(r, \theta)$ the procedure used for determining function $z(r, \theta)$, we obtain

$$u(r, \theta) = \frac{2}{\pi}\int_0^\infty \Lambda_u(\tau, \psi, \theta)\, K_{i\tau}(r)\, d\tau$$

$$\Lambda_u(\tau, \psi, \theta) = \operatorname{sh}[\tau(\pi/2 - \psi)]\operatorname{ch}[\tau(\theta - \psi/2)] / \operatorname{sh}(\psi\tau/2)$$

From this for the height of fluid surface lift at the solid wall we obtain

$$z(r, 0) = z(r, \psi) = 1 + e^{-r\sin\psi} + \frac{2}{\pi}\int_0^\infty \Lambda_u(\tau, \psi, 0)\, K_{i\tau}(r)\, d\tau$$

Finally, taking into accout that for considerable r

$$K_{i\tau}(r) \sim \sqrt{\frac{\pi}{2r}}\, e^{-r} e^{-\pi\tau/2}$$

we determine the behavior of the fluid surface along straight lines parallel to the solid walls: $\lim_{x\to\infty} z(x, y) = e^{-y}$ which means that at some distance from the coordinate origin the fluid surface near to each of the solid walls behaves as for $\psi = \pi$.

Let us show how formula (4.10) can be transformed into (4.5) when $\psi = \pi / n$. To do this we note that [5]

$$\Lambda_z\left(\tau, \frac{\pi}{n}, \theta\right) = \sum_{k=0}^{n-1} \operatorname{ch}\left[\tau\left(\theta + \frac{k\pi}{n} - \frac{\pi}{2}\right)\right]$$

Taking into account that for $0 < \theta < \psi$, and $0 \leqslant k \leqslant n - 1$ we have $|\beta| = |\theta + k\pi/n - \pi/2| < \pi/2$ and that in this case [5]

$$\int_0^\infty \operatorname{ch}(\beta\tau)\, K_{i\tau}(r)\, d\tau = \frac{\pi}{2} e^{-r\cos\beta}$$

we obtain the requisite result

$$z(r, \theta) = \frac{2}{\pi}\sum_{k=0}^{n-1}\int_0^\infty \operatorname{ch}\left[\tau\left(\theta + \frac{k\pi}{n} - \frac{\pi}{2}\right)\right] K_{i\tau}(r)\, d\tau = \sum_{k=0}^{n-1}\exp\left[-r\sin\left(\theta + \frac{k\pi}{n}\right)\right]$$

In concluding we point out the case when the solution of problem (4.3) can be obtained in the form of a sum of series. This happens when Γ represents the sector $0 \leqslant \theta \leqslant \psi$, $0 \leqslant r \leqslant r_0$. We can then write

$$z = -\left(\operatorname{ctg}\frac{\psi}{2}\cos\theta + \sin\theta\right) r + \sum_{n=0}^\infty u_n(r)\cos\left[\frac{2\pi n}{\psi}\left(\theta - \frac{\psi}{2}\right)\right]$$

For functions $u_n(r)$ we obtain equations with boundary conditions

$$u_n'' + \frac{1}{r} u_n' - \left(1 + \frac{4\pi^2 n^2}{\psi^2 r^2}\right) u_n = -A_n r,\ u_0'(r_0) = 1 + A_0,\ u_k'(r_0) = A_k$$
$$A_0 = 2 / \psi,\ A_k = (-1)^{k+1} \psi / (\pi^2 k^2 - \psi^2 / 4),\ k \geqslant 1$$

where A_n are specified coefficients of expansion of function $\operatorname{ctg}(\psi / 2) \cos \theta + \sin \theta$.

From this we obtain for $u_n(r)$ formulas in terms of cylindrical functions.

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UDC 539.3

AXISYMMETRIC PROBLEM OF THE PENETRATION OF A THIN, RIGID, SMOOTH PILE OF FINITE LENGTH INTO AN ELASTIC HALF-SPACE

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The solution of the problem in the title is given in quadratures.

When angular points (for example, a pile with a conucal tip) are present at the section occupied by the pile, tensile stresses are possible near its endpoint if it is assumed that adhesion without friction holds on this section. Otherwise cracks must be taken into account. It has been established that the stresses on the boundary of an axisymmetric pile differ from the corresponding stresses in the plane problem of wedging. Especially simple formulas are obtained in the problem of penetration of semi-infinite pile into an elastic space.

1. Plane problem. The solution of the plane problem of wedging by a thin, rigid, smooth wedge along the ox-axis of an elastic half-space is given in [1]. Let us indicate the results referred here by starting from the representation of the solution as [2]